AIAA JOURNAL
Vol. 35, No. 10, October 1997

Vibration Mode Localization in One-Dimensional Systems

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A general method of regular perturbation for linear eigenvalue problems is presented, in which the orders of perturbation terms are extended to infinity. The method of regular perturbation is employed to study vibration mode localization in randomly disordered weakly coupled one-dimensional cantilever–spring chains. First-order approximate results are obtained for the localization factors, which characterize the average exponential rates of growth or decay of the amplitudes of vibration.

Nomenclature

$\boldsymbol{A}$ = diagonal matrix obtained from the main diagonal of matrix $\bar{\boldsymbol{A}}$
$\bar{\boldsymbol{A}}$ = $N \times N$ matrix
a_{ij}^J, b_{ij}^J = elements of matrix $\bar{\boldsymbol{A}}$ as defined in Eq. (29)
a_{ij}, b_{ij} = underlying random variables for a_{ij}^J, b_{ij}^J
$E[\ ]$ = expected value
EI = expected value of EI_i
EI_i = flexural rigidity of the ith cantilever in the plane of the meshes
$\overline{EI}_i$ = nondimensional parameter defined in Eq. (A10)
H = vertical location of the coupling springs
K_i = stiffness of the spring coupling the ith and the $(i+1)$th cantilevers
$\bar{k}$, $\bar{K}$ = mean values of $\bar{k}_i$, $\bar{K}_i$
k_i = normal stiffness per unit length of the ith mesh in the horizontal direction
$\bar{k}_i$, $\bar{K}_i$ = nondimensional parameters defined in Eq. (A10)
L = length of the cantilevers
m = expected value of m_i
m_i = mass density per unit length of the ith cantilever including the mass lumped from the $(i-1)$th and ith meshes
$\bar{m}_i$ = nondimensional parameter defined in Eq. (A10)
N = number of cantilevers or dimension of $\bar{\boldsymbol{A}}$
S = total number of substructural modes taken for each cantilever
$\boldsymbol{u}_{Jj}$ = vector with all elements being zero except the jth element of the Jth block, which is 1
$\boldsymbol{u}_j$ = vector with all elements being zero except the jth element, which is 1
$\bar{\boldsymbol{u}}_j$ = eigenvalue for the mode in which vibration is originated at the jth cantilever
$\bar{u}_{ji}$ = ith element of vector $\bar{\boldsymbol{u}}_j$
$v_i(y, t)$ = in-plane deflection of the ith cantilever
x_{ij} = generalized coordinates
α = underlying random variable for α_i
α_i = ith diagonal element of $\bar{\boldsymbol{A}}$ or $\boldsymbol{A}$
$\hat{\alpha}_i$ = parameter defined in Eq. (A4)
β = underlying random variable for β_i
β_i = ith off-diagonal element of the tridiagonal matrix $\bar{\boldsymbol{A}}$
$\hat{\beta}_i$ = parameter defined in Eq. (A4)
δ_{ij} = Kronecker delta symbol
$\delta \boldsymbol{A}$ = ith-order perturbation of matrix $\bar{\boldsymbol{A}}$
$\delta \nu_j$ = ith-order perturbation of $\bar{\nu}_j$, $\delta^0 \nu_j = \nu_j$
$\delta \nu_{Jj}$ = ith-order perturbation of $\bar{\nu}_{Jj}$
$\delta \boldsymbol{u}_j$ = ith-order perturbation of $\bar{\boldsymbol{u}}_j$, $\delta^0 \boldsymbol{u}_j = \boldsymbol{u}_j$
δ_X = coefficient of variation of the random variable X
$\varepsilon_{Ij,\,Ii}^M$ = amplitude of vibration of the Ith cantilever in the ith substructural mode in the Mth-order perturbation for the mode in which vibration is originated at the Jth cantilever in the jth substructural mode
$\varepsilon_{j,\,k}^M$ = kth element of vector $\delta^M \boldsymbol{u}_j$
λ_j = localization factor of the mode in which vibration is originated at the jth cantilever
μ_X = mean value of the random variable X
ν = underlying random variable for ν_i
$\bar{\nu}_{Jj}$ = eigenvalue of the mode in which vibration is originated at the Jth cantilever in the jth substructural mode
ν_j = unperturbed value of $\bar{\nu}_j$
$\bar{\nu}_j$ = eigenvalue for the mode in which vibration is originated at the jth cantilever
σ_X = standard deviation of the random variable X
ϕ_j = jth normal mode of the transverse vibration of a cantilever, Eq. (A4)

I. Introduction

BECAUSE of defects in manufacturing and assembly, no structures designed as a periodic structure can be perfectly periodic. Disorder can occur in the geometry of configuration and material properties of the structure. The dynamical behavior of a disordered periodic system could be significantly different from that of a perfectly periodic system. For a perfectly periodic structure, the vibration modes are of wavy shapes and extend throughout the structure, whereas when the structure is disordered, vibration is confined to a small region, with the amplitudes of vibration decaying exponentially away from a center. This phenomenon is known as vibration mode localization. The average exponential rates at which the vibration amplitudes decay are the localization factors. It is therefore of practical importance to determine the localization factors.

Since the pioneering work of Anderson on localization in disordered periodic systems in the field of solid state physics, much work has been done during the past three decades. Several theoretical methods have been applied to study the localization phenomena and to determine the localization factors. Ishii[1] studied localization in one-dimensional systems by applying Furstenberg's theorem[2] on the limiting behavior of products of random matrices. Herbert and Jones[3] and Thouless[4] employed a Green's function formulation to study the same systems and obtained expressions for the localization factors in terms of the natural frequencies of the systems.

In structural dynamics, Hodges[5] was the first to recognize the relevance of localization theory to dynamical behavior of periodic engineering structures. Pierre and Dowell[6] studied the localization phenomenon for disordered structural systems consisting of weakly coupled component systems using a perturbation method. It was found that the sensitivity to disorder depends primarily on the ratio of disorder strength to coupling strength. For a small disorder-to-coupling ratio, i.e., strong coupling, the disordered structure behaved like a perturbation of the corresponding perfect structure and disorder had a relatively small effect. For a large disorder-to-coupling ratio, i.e., weak coupling, the disordered structure behaved

Received Dec. 5, 1996; revision received July 10, 1997; accepted for publication July 15, 1997.

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like a perturbation of the corresponding decoupled disordered structure, and disorder dramatically changed the dynamics of the system. Pierre[7] studied the localization problem of vibration propagation from a source of excitation in a mono-coupled cantilever–spring chain using the method of perturbation. Cha and Pierre[8] studied the transmission of steady-state harmonic vibration from a local source of excitation in a mono-coupled cantilever–spring chain with multicomponent mode substructures. A transfer matrix formulation was used to derive first-order approximation of the localization factors for strong and weak modal coupling.

Kissel[9] and Ariaratnam and Xie[10] used a traveling wave approach to investigate the localization effects on one-dimensional periodic engineering structures. A transfer matrix formulation including wave transfer matrices was used to model disordered periodic structures. Furstenberg's theorem for products of random matrices was applied to calculate the localization factors as a function of frequency. Cai and Lin[11] developed a perturbation scheme to calculate the localization factor based on a generic periodic structure.

Using the multiplicative ergodic theorem of Oseledec,[12] Kissel[13] derived the localization factors as a function of the transmission matrix for multiwave disordered systems. The localization factor is related to the smallest positive Lyapunov exponent of the system. Xie and Ariaratnam[14, 15] studied vibration mode localization in disordered cyclic structures with both single and multiple substructural modes. The localization factors were determined by the method of transfer matrix and the method of Green's function.

The method of perturbation has been employed extensively to study localization in disordered systems. However, in all of these applications, only a small number of perturbation terms is usually taken. In this paper, the method of regular perturbation is extended to include perturbation terms of infinite orders. The free vibration of weakly mono-coupled and multicoupled one-dimensional disordered systems is considered. First-order approximate asymptotic results for the localization factors are obtained using the method of regular perturbation.

Studies on the localization phenomenon in structural dynamics have been restricted to one-dimensional structures. Because of the degree of difficulty and the amount of computation involved in studying higher-dimensional systems, no research work on the localization in these systems has been published in the context of structural dynamics. These large higher-dimensional periodic structures, such as two-dimensional planar lattice trusses, have important applications in space station platforms. Localization in randomly disordered higher-dimensional periodic structures is of both theoretical and practical importance. Employing the method of perturbation presented in this paper, vibration mode localization in two-dimensional disordered cantilever–spring arrays is studied in the companion paper.[16]

II. Method of Perturbation for Linear Eigenvalue Problems

The variations of the parameters of randomly disordered systems can be considered as perturbations applied to the systems. Therefore the method of perturbation can be employed to analyze localization in disordered systems. Perturbation analysis is efficient as compared with a global eigenvalue analysis of the entire system and provides physical insight into the localization phenomena. In this section, a general method of regular perturbation for linear eigenvalue problems is presented.

Consider a linear eigenvalue problem

$$\bar{\boldsymbol{A}}\bar{\boldsymbol{u}}_j = \bar{\nu}_j \bar{\boldsymbol{u}}_j, \qquad j = 1, 2, \ldots, N \tag{1}$$

If the main diagonal elements of $\bar{\boldsymbol{A}}$ are much larger than the off-diagonal elements in magnitude, $\bar{\boldsymbol{A}}$ may be written as

$$\bar{\boldsymbol{A}} = \boldsymbol{A} + \delta\boldsymbol{A} + \ldots + \delta^i \boldsymbol{A} + \ldots \tag{2}$$

where

$$\boldsymbol{A} = \text{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_N\} \tag{3}$$

The eigenvalues and eigenvectors of $\bar{\boldsymbol{A}}$ may be written as, for $j = 1, 2, \ldots, N$,

$$\begin{aligned} \bar{\nu}_j &= \nu_j + \delta\nu_j + \ldots + \delta^i \nu_j + \ldots \\ \bar{\boldsymbol{u}}_j &= \boldsymbol{u}_j + \delta\boldsymbol{u}_j + \ldots + \delta^i \boldsymbol{u}_j + \ldots \end{aligned} \tag{4}$$

Once one substitutes Eqs. (2) and (4) into Eq. (1), expands, and collects terms of the same order, the Mth-order terms satisfy the following equation:

$$\sum_{m=0}^{M} (\delta^{M-m}\boldsymbol{A}\delta^m \boldsymbol{u}_j) = \sum_{m=0}^{M} (\delta^{M-m}\nu_j \delta^m \boldsymbol{u}_j), \qquad j = 1, 2, \ldots, N \tag{5}$$

For the unperturbed system, i.e., $M = 0$, Eq. (5) becomes

$$\boldsymbol{A}\boldsymbol{u}_j = \nu_j \boldsymbol{u}_j, \qquad j = 1, 2, \ldots, N \tag{6}$$

Equation (6) is the unperturbed eigenvalue problem with eigenvalue $\nu_j = \alpha_j$ and the corresponding eigenvector $\boldsymbol{u}_j$.

For $M \geqslant 1$, Eq. (5) may be written as, for $j = 1, 2, \ldots, N$,

$$\begin{aligned} (\boldsymbol{A} - \nu_j \boldsymbol{I})\delta^M \boldsymbol{u}_j - \delta^M \nu_j \boldsymbol{u}_j &= \sum_{m=0}^{M-1} (\delta^{M-m}\nu_j \delta^m \boldsymbol{u}_j) \\ &\quad - \sum_{m=0}^{M-1} (\delta^{M-m}\boldsymbol{A}\delta^m \boldsymbol{u}_j) \end{aligned} \tag{7}$$

The mth-order perturbation of eigenvector $\delta^m \boldsymbol{u}_j$, $m = 1, 2, \ldots, M$, may be expressed as a linear combination of the unperturbed eigenvectors as

$$\delta^m \boldsymbol{u}_j = \sum_{k=1}^{N} \varepsilon_{j,k}^m \boldsymbol{u}_k, \qquad j = 1, 2, \ldots, N \tag{8}$$

where $\varepsilon_{j,k}^0 = \delta_{j,k}$.

Substituting Eq. (8) into Eq. (7) and multiplying Eq. (7) by $\boldsymbol{u}_i^T$ from the left results in, for $M \geqslant 1$ and $j = 1, 2, \ldots, N$,

$$\begin{aligned} \boldsymbol{u}_i^T (\boldsymbol{A} - \nu_j \boldsymbol{I}) \sum_{k=1}^{N} \varepsilon_{j,k}^M \boldsymbol{u}_k - \boldsymbol{u}_i^T \delta^M \nu_j \boldsymbol{u}_j \\ = \boldsymbol{u}_i^T \left[\sum_{m=0}^{M-1} (\delta^{M-m}\nu_j \delta^m \boldsymbol{u}_j) - \sum_{m=0}^{M-1} (\delta^{M-m}\boldsymbol{A}\delta^m \boldsymbol{u}_j) \right] \end{aligned} \tag{9}$$

which may be simplified as

$$\begin{aligned} \varepsilon_{j,i}^M (\nu_i - \nu_j) - \delta^M \nu_j \delta_{i,j} \\ = \boldsymbol{u}_i^T \left[\sum_{m=0}^{M-1} (\delta^{M-m}\nu_j \delta^m \boldsymbol{u}_j) - \sum_{m=0}^{M-1} (\delta^{M-m}\boldsymbol{A}\delta^m \boldsymbol{u}_j) \right] \end{aligned} \tag{10}$$

If $i = j$, Eq. (10) becomes, for $M \geqslant 1$,

$$\delta^M \nu_j = \boldsymbol{u}_j^T \left[\sum_{m=0}^{M-1} (\delta^{M-m}\boldsymbol{A}\delta^m \boldsymbol{u}_j) - \sum_{m=0}^{M-1} (\delta^{M-m}\nu_j \delta^m \boldsymbol{u}_j) \right] \tag{11}$$

If $i \neq j$, Eq. (10) leads to, for $M \geqslant 1$,

$$\varepsilon_{j,i}^M = \frac{\boldsymbol{u}_i^T \left[\sum_{m=0}^{M-1} (\delta^{M-m}\boldsymbol{A}\delta^m \boldsymbol{u}_j) - \sum_{m=1}^{M-1} (\delta^{M-m}\nu_j \delta^m \boldsymbol{u}_j) \right]}{\nu_j - \nu_i} \tag{12}$$

Substituting Eq. (8) into Eqs. (11) and (12) results in

$$\varepsilon_{j,i}^M = \frac{\sum_{m=0}^{M-1} \sum_{k=1}^{N} \varepsilon_{j,k}^m (\boldsymbol{u}_i^T \delta^{M-m} \boldsymbol{A}\boldsymbol{u}_k) - \sum_{m=1}^{M-1} \delta^{M-m}\nu_j \varepsilon_{j,i}^m}{\nu_j - \nu_i} \qquad i \neq j \tag{13}$$

$$\delta^M \nu_j = \sum_{m=0}^{M-1} \sum_{k=1}^{N} \varepsilon_{j,k}^m (\boldsymbol{u}_j^T \delta^{M-m} \boldsymbol{A} \boldsymbol{u}_k) - \sum_{m=1}^{M-1} \delta^{M-m} \nu_j \varepsilon_{j,j}^m \tag{14}$$

The coefficient $\varepsilon_{j,j}^M$ is determined by normalizing the eigenvector $\bar{\boldsymbol{u}}_j$, i.e., $\boldsymbol{u}_j^T \bar{\boldsymbol{u}}_j = 1$, and may be shown to be given by

$$\varepsilon_{j,j}^M = -\frac{1}{2} \sum_{k=1}^{N} \sum_{m=0}^{M-1} \varepsilon_{j,k}^m \varepsilon_{j,k}^{M-m} \tag{15}$$

If $\bar{\boldsymbol{A}} = \boldsymbol{A} + \delta \boldsymbol{A}$, Eqs. (13) and (14) reduce to, for $M \geqslant 1$,

$$\varepsilon_{j,i}^M = \frac{\sum_{k=1}^{N} \varepsilon_{j,k}^{M-1} (\boldsymbol{u}_i^T \delta \boldsymbol{A} \boldsymbol{u}_k) - \sum_{m=1}^{M-1} \delta^{M-m} \nu_j \varepsilon_{j,i}^m}{\nu_j - \nu_i}, \qquad i \neq j \tag{16}$$

$$\delta^M \nu_j = \sum_{k=1}^{N} \varepsilon_{j,k}^{M-1} (\boldsymbol{u}_j^T \delta \boldsymbol{A} \boldsymbol{u}_k) - \sum_{m=1}^{M-1} \delta^{M-m} \nu_j \varepsilon_{j,j}^m \tag{17}$$

The general formulations of regular perturbation for a linear eigenvalue problem derived in this section will be employed to investigate vibration mode localization in weakly coupled randomly disordered one-dimensional cantilever–spring chains in the following sections.

The method of regular perturbation presented here has some limitations. A major drawback of the method is that it can produce secular terms. In each $\varepsilon_{j,i}^M$ term, there is a term $(\nu_j - \nu_i)$ appearing in the denominator, which means that, when the eigenvalues of the unperturbed system are nearly identical, the results presented are not valid. To overcome this problem, researchers have used other methods such as the method of matched asymptotic expansion.[17] However, these results are applicable only to matrix $\boldsymbol{A}$ with small values of N.

Therefore, the localization analysis performed in this study is restricted to small ratios of perturbation (coupling) to disorder in α (diagonal elements of matrix $\bar{\boldsymbol{A}}$) such that the terms $\varepsilon_{j,i}^M$ given by Eq. (13) or (16) remain as small perturbations. The cases of larger ratios of perturbation to disorder in α, i.e., when the terms $\varepsilon_{j,i}^M$ given by Eq. (13) or (16) cannot be regarded as small perturbations, will be studied in future research.

III. Vibration Mode Localization in Randomly Disordered, Mono-Coupled Cantilever–Spring Chains

For clarity of presentation, consider first the case when the cantilever–spring chain is mono-coupled, which is the result when only one cantilever (substructural) mode is taken for each cantilever. As derived in the Appendix, the equation of free vibration of a cantilever–spring chain is given by a linear eigenvalue problem (A9):

$$\bar{\boldsymbol{A}} \bar{\boldsymbol{u}} = \bar{\nu} \bar{\boldsymbol{u}} \tag{18}$$

where $\bar{\boldsymbol{A}}$ is a symmetric tridiagonal matrix given by Eq. (A11). When the coupling between any two adjacent cantilevers is weak, namely, when $\bar{\beta}_j^4 \gg \bar{k} + \bar{K} \phi_j^2(H)$, where j is the cantilever mode considered, matrix $\bar{\boldsymbol{A}}$ may be written as $\boldsymbol{A} = \boldsymbol{A} + \delta \boldsymbol{A}$, where $\boldsymbol{A} = \text{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ and

$$\delta \boldsymbol{A} = \begin{bmatrix} 0 & \beta_1 & & & \\ \beta_1 & 0 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{N-2} & 0 & \beta_{N-1} \\ & & & \beta_{N-1} & 0 \end{bmatrix}$$

For the vibration mode in which vibration is originated at the jth cantilever, the eigenvalue $\bar{\nu}_j$ and the corresponding eigenvector $\bar{\boldsymbol{u}}_j$ may be expressed in the form of Eq. (4). When one applies the regular perturbation procedure presented in Sec. II, the coefficients $\varepsilon_{j,i}^M$ of the Mth-order perturbation are given by, from Eq. (16),

$$\varepsilon_{j,i}^M = \frac{\beta_{i-1} \varepsilon_{j,i-1}^{M-1} + \beta_i \varepsilon_{j,i+1}^{M-1} - \sum_{m=1}^{M-1} \delta^{M-m} \nu_j \varepsilon_{j,i}^m}{\nu_j - \nu_i}$$

$$i \neq j, \qquad i = 1, 2, \ldots, N \tag{19}$$

The Mth-order perturbation of eigenvalue $\delta^M \nu_j$ is given by Eq. (17) and is simplified as follows:

$$\delta^M \nu_j = \beta_{j-1} \varepsilon_{j,j-1}^{M-1} + \beta_j \varepsilon_{j,j+1}^{M-1} - \sum_{m=1}^{M-1} \delta^{M-m} \nu_j \varepsilon_{j,j}^m \tag{20}$$

Because vibration is originated at the jth cantilever, for $M = 0$, $\varepsilon_{j,j}^0 = 1$, $\varepsilon_{j,i}^0 = 0$, for $i \neq j$, and $\delta^0 \nu_j = \nu_j = \alpha_j$. Using the method of mathematical induction, it may be shown that, for all positive integers M,

$$\begin{aligned} &\varepsilon_{j,j \pm k}^M = 0, \qquad \text{for} \quad k + M \text{ odd}, \qquad k \leqslant M \\ &\delta^M \nu_j = 0, \qquad \text{for} \quad M \text{ odd} \end{aligned} \tag{21}$$

Furthermore, it is obvious that $\varepsilon_{j,j \pm k}^M = 0$ for all $k > M$. For $\varepsilon_{j,j}^M$, substituting the first of Eqs. (21) into Eq. (15) leads to

$$\varepsilon_{j,j}^M = \begin{cases} 0, & \text{for} \quad M \text{ odd} \\ -\frac{1}{2} \sum_{k=1}^{N} \sum_{m=0}^{M-1} \varepsilon_{j,k}^m \varepsilon_{j,k}^{M-m}, & \text{for} \quad M \text{ even} \end{cases} \tag{22}$$

Therefore, for the vibration mode in which vibration is originated at the jth cantilever, the extension or "propagation" of vibration is as shown in Fig. 1, which is the result of the extension of nonzero terms of $\varepsilon_{j,j \pm k}^M$ to large values of k as M increases. The arrows indicate the dependence of the higher-order perturbation terms on the lower-order perturbation terms. For $M = 0$, only the jth cantilever is vibrating, whereas for $M = 1$, three cantilevers $(j-1, j, j+1)$ are vibrating. In general, for the Mth-order perturbation, vibration (perturbation) is extended to the $(j \pm M)$th cantilevers; hence there are $2M + 1$ cantilevers vibrating.

Note that $\varepsilon_{j,j+M}^M$ depends only on $\varepsilon_{j,j+M-1}^{M-1}$. In fact, from Eq. (19),

$$\begin{aligned} \varepsilon_{j,j+M}^M &= \frac{\beta_{j+M-1}}{\nu_j - \nu_{j+M}} \cdot \varepsilon_{j,j+M-1}^{M-1} \\ &= \frac{\beta_{j+M-1}}{\nu_j - \nu_{j+M}} \cdot \frac{\beta_{j+M-2}}{\nu_j - \nu_{j+M-1}} \cdots \frac{\beta_j}{\nu_j - \nu_{j+1}} \cdot \varepsilon_{j,j}^0 \end{aligned} \tag{23}$$

Equation (23) may be used to determine a first-order approximation of the localization factor of the corresponding vibration mode, which characterizes the average exponential rate of decay of the amplitudes

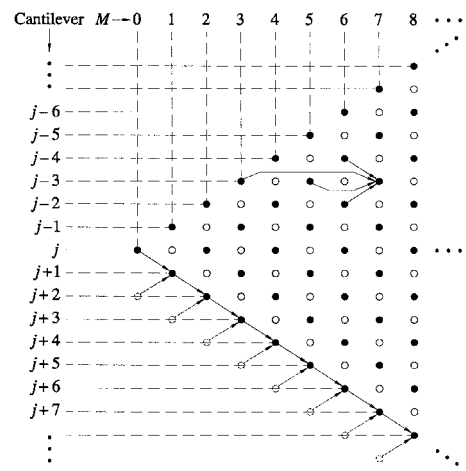


Fig. 1 Extension of vibration of a mono-coupled one-dimensional system: ○ zero terms and ● nonzero terms.

of vibration. For the vibration mode in which vibration is originated at the jth cantilever, the localization factor is defined as

$$\lambda_j = \lim_{M\to\infty} -\frac{1}{M} \ln \frac{|\bar{u}_{j(j+M)}|}{|\bar{u}_{jj}|}$$

For a first-order approximation, taking only the leading terms or letting $\bar{u}_{j(j+M)} = \varepsilon^M_{j,j+M}$ leads to

$$\lambda^{(1)}_j = \lim_{M\to\infty} -(1/M) \ln |\varepsilon^M_{j,j+M}| \tag{24}$$

Hence, because $\nu_i = \alpha_i$ for all i, substituting Eq. (23) into Eq. (24) results in

$$\lambda^{(1)}_j = \lim_{M\to\infty} \frac{1}{M} \sum_{m}^{M} \ln|\alpha_j - \alpha_{j+m}| - \lim_{M\to\infty} \frac{1}{M} \sum_{m}^{M} \ln|\beta_{j+m-1}| \tag{25}$$

Using the definition of the expected value, Eq. (25) becomes

$$\lambda^{(1)}_j = E[\ln|\alpha_j - \alpha|] - E[\ln|\beta|] \tag{25$'$}$$

For the special case when α and β are uniformly distributed random variables with means μ_α and μ_β and standard deviations σ_α and σ_β, respectively, a simple analytical result for the localization factor $\lambda^{(1)}_j$ may be obtained as

$$\lambda^{(1)}_j = 1/(2\sqrt{3})\{(1/\sigma_\alpha)[(\alpha_j - a_\alpha)\ln|\alpha_j - a_\alpha| - (\alpha_j - b_\alpha) \times \ln|\alpha_j - b_\alpha|] - (1/\sigma_\beta)(b_\beta \ln|b_\beta| - a_\beta \ln|a_\beta|)\} \tag{26}$$

where $a_\alpha = \mu_\alpha - \sqrt{3}\sigma_\alpha$, $b_\alpha = \mu_\alpha + \sqrt{3}\sigma_\alpha$, $a_\beta = \mu_\beta - \sqrt{3}\sigma_\beta$, and $b_\beta = \mu_\beta + \sqrt{3}\sigma_\beta$. The validity of Eqs. (25) and (26) depends on the convergence of Eq. (23).

Note that, because only the leading term $\varepsilon^M_{j,j+M}$ is used in determining the localization factor λ_j, the results given by Eqs. (24–26) are only first-order approximations of the localization factors.

Herbert and Jones[3] and Thouless[4] employed a Green's function formulation to study system (18) of dimension N and obtained the exact expression for the localization factor of the mode corresponding to the nondimensional natural frequency $\bar{\nu}_j$:

$$\lambda_j = \lim_{N\to\infty} \frac{1}{N-1} \left(\sum_{i\neq j}^{N} \ln|\bar{\nu}_j - \bar{\nu}_i| - \sum_{i}^{N-1} \ln|\beta_i| \right) \qquad j = 1, 2, \ldots, N \tag{27}$$

$$= E[\ln|\bar{\nu}_j - \bar{\nu}|] - E[\ln|\beta|] \tag{27$'$}$$

(see Ref. 14 for a more detailed presentation). Equation (25) may be easily obtained from Eq. (27). It is known that the zeroth-order approximation of $\bar{\nu}_i$ is $\nu_i = \alpha_i$ and the first-order correction $\delta\nu_i = 0$. Hence, the first-order approximation of $\bar{\nu}_i$ is $\bar{\nu}^{(1)}_i = \alpha_i$. Substituting $\bar{\nu}^{(1)}_i$ into Eq. (27) yields

$$\lambda^{(1)}_j = \lim_{N\to\infty} \frac{1}{N-1} \left[\sum_{i\neq j}^{N} \ln|\bar{\nu}^{(1)}_j - \bar{\nu}^{(1)}_i| - \sum_{i}^{N-1} \ln|\beta_i| \right]$$

$$= \lim_{N\to\infty} \frac{1}{N-1} \left(\sum_{i\neq j}^{N} \ln|\alpha_j - \alpha_i| - \sum_{i}^{N-1} \ln|\beta_i| \right) \tag{28}$$

which is equivalent to Eq. (25) or (25′).

Although Eq. (24) is only a first-order approximation of the localization factor, the process of deriving Eq. (24) using the method of regular perturbation provides some insights into the phenomenon of vibration mode localization in weakly coupled systems.

Similar results as Eqs. (25) and (26) have been obtained by Pierre[7] for vibration propagation in weakly mono-coupled one-dimensional cantilever–spring chains.

In Ref. 14, two numerical schemes were presented for evaluating the localization factors of system (18), namely, the method using a transfer matrix formulation and the method involving an invariant probability density on a semicircle. Both schemes may be implemented easily, and the results are considered as numerically exact for a large number of iterations N.

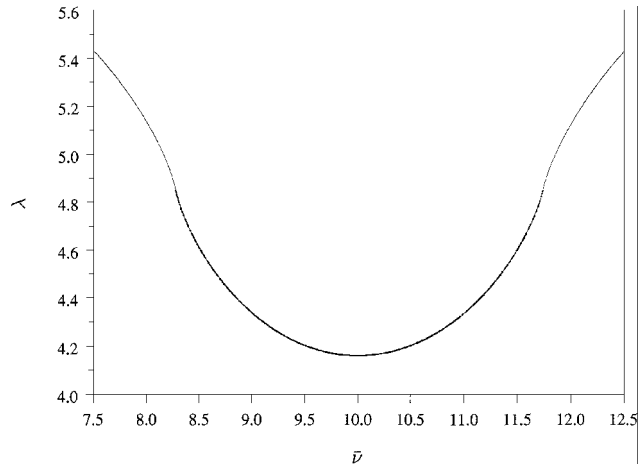


Fig. 2 Localization factors of a mono-coupled one-dimensional system: $\mu_\alpha = 10$, $\delta_\alpha = 0.1$, $\mu_\beta = -0.01$, and $\delta_\beta = 0.1$; ——, transfer matrix method, $N = 10^7$; ●, perturbation method, $N = 10^3$ and $M = 20$.

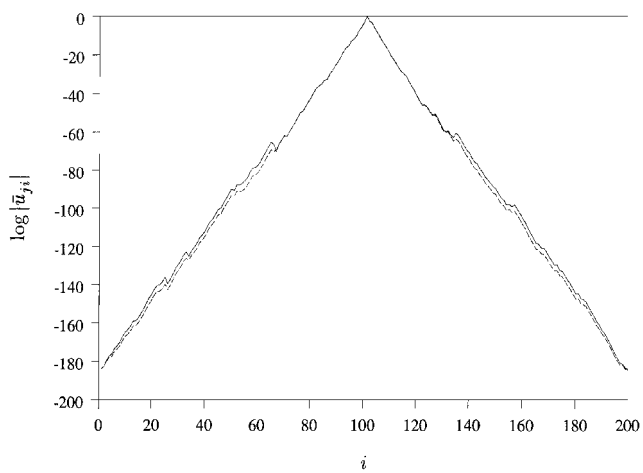


Fig. 3 Amplitudes of vibration of a mono-coupled one-dimensional system: $\mu_\alpha = 10$, $\delta_\alpha = 0.1$, $\mu_\beta = -0.01$, $\delta_\beta = 0.1$, $M = 100$, $N = 2M + 1$, $j = M + 1$, $\alpha_j = 9.9960$, $\bar{\nu}_j = 9.9946$, and $\lambda_j = 1.8522$; ——, exact (using all perturbation terms); – – –, approximate (using only end terms $\varepsilon^m_{j,j+m}$).

In Fig. 2, numerical results for localization factors given by Eq. (26) for a system with $\mu_\alpha = 10$, $\delta_\alpha = \sigma_\alpha/|\mu_\alpha| = 0.1$, $\mu_\beta = -0.01$, $\delta_\beta = \sigma_\beta/|\mu_\beta| = 0.1$, and $N = 1000$ are compared with numerical results obtained using the method of transfer matrix for $N = 10^7$. In numerical simulation, a set of uniformly distributed random numbers α_j, β_j, $j = 1, 2, \ldots, N$, are generated, and Eq. (26) is used to determine the localization factor λ_j of the mode in which vibration is originated at the jth cantilever. The corresponding nondimensional natural frequency $\bar{\nu}_j$ is determined using Eqs. (4) and (20). It is seen that the first-order approximate results given by Eq. (26) agree very well with the numerical results obtained using the method of transfer matrix.

A vibration mode is shown in Fig. 3 to illustrate localization and exponential decay (or growth) of the amplitudes of vibration. The system consists of 201 cantilevers, and vibration is originated at the 100th cantilever. The solid line represents the exact eigenvector $\bar{\boldsymbol{u}}_j$ obtained using Eqs. (4) and (19) for $M = 100$, whereas the dashed line shows only the end terms in the expression, i.e., $\varepsilon^m_{j,j+m}$, $m = 1, 2, \ldots, 100$.

IV. Vibration Mode Localization in Randomly Disordered Multicoupled Cantilever–Spring Chains

As derived in the Appendix, the equation of free vibration of the multi-coupled cantilever–spring chain as shown in Fig. A1, in which S substructural modes are taken for each cantilever, is given by a linear eigenvalue problem (A9), where $\bar{\boldsymbol{A}}$ is a symmetric block

tridiagonal matrix given by Eq. (A10). For weak coupling, matrix $\bar{\boldsymbol{A}}$ may be written as $\bar{\boldsymbol{A}} = \boldsymbol{A} + \delta\boldsymbol{A}$, where

$$\boldsymbol{A} = \operatorname{diag}\{\boldsymbol{A}_1^0, \boldsymbol{A}_2^0, \ldots, \boldsymbol{A}_N^0\}, \qquad \boldsymbol{A}_J^0 = \operatorname{diag}\{a_{11}^J, a_{22}^J, \ldots, a_{SS}^J\}$$

$$\delta\boldsymbol{A} = \begin{bmatrix} \hat{\boldsymbol{A}}_1 & \boldsymbol{B}_1 & & & \\ \boldsymbol{B}_1 & \hat{\boldsymbol{A}}_2 & \boldsymbol{B}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \boldsymbol{B}_{N-2} & \hat{\boldsymbol{A}}_{N-1} & \boldsymbol{B}_{N-1} \\ & & & \boldsymbol{B}_{N-1} & \hat{\boldsymbol{A}}_N \end{bmatrix} \tag{29}$$

$$\hat{\boldsymbol{A}}_J = \begin{bmatrix} 0 & a_{12}^J & a_{13}^J & \cdots & a_{1,S-1}^J & a_{1S}^J \\ a_{21}^J & 0 & a_{23}^J & \cdots & a_{2,S-1}^J & a_{2S}^J \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{S1}^J & a_{S2}^J & a_{S3}^J & \cdots & a_{S,S-1}^J & 0 \end{bmatrix}, \qquad \boldsymbol{B}_J = [b_{kl}^J]$$

From symmetry of matrix $\bar{\boldsymbol{A}}$, $a_{kl}^J = a_{lk}^J$, $b_{kl}^J = b_{lk}^J$, and the value of each element is given in the Appendix. The eigenvector is

$$\bar{\boldsymbol{u}} = \begin{Bmatrix} \bar{\boldsymbol{u}}_1 \\ \bar{\boldsymbol{u}}_2 \\ \vdots \\ \bar{\boldsymbol{u}}_N \end{Bmatrix}, \qquad \bar{\boldsymbol{u}}_J = \begin{Bmatrix} u_{J1} \\ u_{J2} \\ \vdots \\ u_{JS} \end{Bmatrix}, \qquad J = 1, 2, \ldots, N$$

Because S substructural modes are considered for each cantilever, there are S distinct frequency groups, each of which corresponds to a cantilever mode.

For the vibration mode in which vibration is originated at the Jth cantilever in the jth substructural mode, the unperturbed eigenvalue is $\nu_{Jj} = a_{j,j}^J$ and the corresponding eigenvector is $\boldsymbol{u}_{Jj}$. Following the regular perturbation procedure presented in Sec. II, the coefficient $\varepsilon_{Jj,Ii}^M$ is given by, from Eq. (16),

$$\begin{aligned} \varepsilon_{Jj,Ii}^M &= \frac{\boldsymbol{u}_{Ii}^T(\delta\boldsymbol{A}\delta^{M-1}\boldsymbol{u}_{Jj} - \sum_{m=1}^{M-1}\delta^{M-m}\nu_{Jj}\delta^m\boldsymbol{u}_{Jj})}{\nu_{Jj} - \nu_{Ii}} \\ &= \frac{\sum_{K=1}^{N}\sum_{k=1}^{S}\varepsilon_{Jj,Kk}^{M-1}\boldsymbol{u}_{Ii}^T\delta\boldsymbol{A}\boldsymbol{u}_{Kk} - \sum_{m=1}^{M}\boldsymbol{u}_{Ii}^T\delta^{M-m}\nu_{Jj}\sum_{K=1}^{N}\sum_{k=1}^{S}\varepsilon_{Jj,Kk}^m\boldsymbol{u}_{Kk}}{\nu_{Jj} - \nu_{Ii}} \\ &= \frac{\sum_{K=1}^{N}\sum_{k=1}^{S}\varepsilon_{Jj,Kk}^{M-1}(b_{ki}^{K-1}\delta_{I,K-1} + a_{ik}^K\delta_{I,K} + b_{ik}^K\delta_{I,K+1}) - \sum_{m=1}^{M-1}\sum_{K=1}^{N}\sum_{k=1}^{S}\varepsilon_{Jj,Kk}^m\delta^{M-m}\nu_{Jj}\delta_{Ii,Kk}}{\nu_{Jj} - \nu_{Ii}} \\ &= \frac{\sum_{k=1}^{S}\varepsilon_{Jj,(I+1)k}^{M-1}b_{ki}^I + \sum_{\substack{k=1 \\ k\neq i}}^{S}\varepsilon_{Jj,Ik}^{M-1}a_{ik}^I + \sum_{k=1}^{S}\varepsilon_{Jj,(I-1)k}^{M-1}b_{ik}^{I-1} - \sum_{m=1}^{M-1}\varepsilon_{Jj,Ii}^m\delta^{M-m}\nu_{Jj}}{\nu_{Jj} - \nu_{Ii}} \end{aligned}$$

Using the same argument as presented in Sec. III, for the $(M-1)$th perturbation, the amplitudes of vibration of the $(J \pm M)$th cantilevers are zero in all substructural modes, whereas for the Mth-order perturbation, the amplitudes of vibration of the $(J \pm M)$th cantilevers depend only on those of the $(J \pm M \mp 1)$th cantilevers, respectively. Hence, for $i = 1, 2, \ldots, S$,

$$\begin{aligned} \varepsilon_{Jj,(J+M)i}^M &= \frac{\sum_{k=1}^{S}\varepsilon_{Jj,(J+M-1)k}^{M-1}b_{ik}^{J+M-1}}{\nu_{Jj} - \nu_{(J+M)i}} \\ &= \frac{1}{\nu_{Jj} - \nu_{(J+M)i}}\{b_{i1}^{J+M-1} \quad b_{i2}^{J+M-1} \quad \cdots b_{iS}^{J+M-1}\} \\ &\quad \times \begin{Bmatrix} \varepsilon_{Jj,(J+M-1)1}^{M-1} \\ \varepsilon_{Jj,(J+M-1)2}^{M-1} \\ \vdots \\ \varepsilon_{Jj,(J+M-1)S}^{M-1} \end{Bmatrix} \end{aligned} \tag{30}$$

Because vibration is originated at the Jth cantilever in the jth substructural mode, the natural frequency of the system is in the jth frequency group and the jth substructural mode is dominant for each cantilever. If the frequency groups or the passbands are well separated, the term $1/[\nu_{Jj} - \nu_{(J+M)j}]$ is much larger than $1/[\nu_{Jj} - \nu_{(J+M)i}]$, and $\varepsilon_{Jj,(J+M)j}^M$ is much larger than $\varepsilon_{Jj,(J+M)i}^M$, $i = 1, 2, \ldots, S$, $i \neq j$. Therefore,

$$\varepsilon_{Jj,(J+M)j}^M \approx \frac{b_{jj}^{J+M-1}}{\nu_{Jj} - \nu_{(J+M)j}}\varepsilon_{Jj,(J+M-1)j}^{M-1} \tag{31}$$

and one may take the norm of vector

$$\boldsymbol{\varepsilon}_{J+M}^M = \{\varepsilon_{Jj,(J+M)1}^M, \varepsilon_{Jj,(J+M)2}^M, \ldots, \varepsilon_{Jj,(J+M)S}^M\}^T$$

as

$$\begin{aligned} \|\boldsymbol{\varepsilon}_{J+M}^M\| &= \|\boldsymbol{\varepsilon}_{J+M}^M\|_\infty \\ &= \max\{|\varepsilon_{Jj,(J+M)1}^M|, |\varepsilon_{Jj,(J+M)2}^M|, \ldots, |\varepsilon_{Jj,(J+M)S}^M|\} \\ &= |\varepsilon_{Jj,(J+M)j}^M| \end{aligned} \tag{32}$$

From Eqs. (31) and (32),

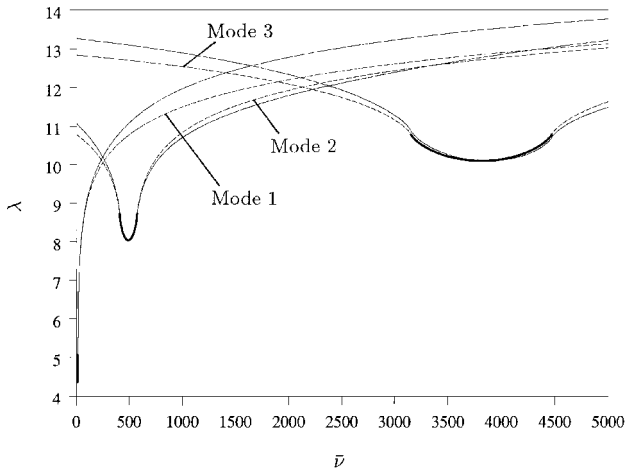
$$\begin{aligned} &\|\boldsymbol{\varepsilon}_{J+M}^M\| \\ &\quad \approx \left|\frac{b_{jj}^{J+M-1}}{\nu_{Jj} - \nu_{(J+M)j}}\right| \|\boldsymbol{\varepsilon}_{J+M-1}^{M-1}\| \\ &\quad \approx \left|\frac{b_{jj}^{J+M-1}}{\nu_{Jj} - \nu_{(J+M)j}} \cdot \frac{b_{jj}^{J+M-2}}{\nu_{Jj} - \nu_{(J+M-1)j}} \cdots \frac{b_{jj}^{J}}{\nu_{Jj} - \nu_{(J+1)j}}\right| \|\boldsymbol{\varepsilon}_J^0\| \end{aligned} \tag{33}$$

A first-order localization factor for the vibration mode in which vibration is originated at the Jth cantilever in the jth substructural mode is

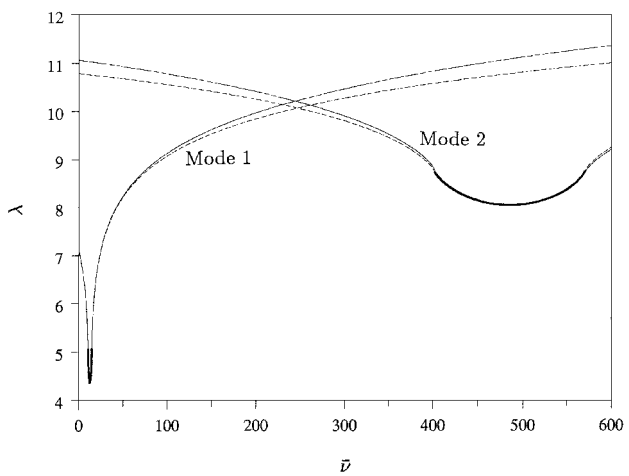
$$\begin{aligned} \lambda_{Jj}^{(1)} &= \lim_{M\to\infty} -\frac{1}{M}\ln\|\boldsymbol{\varepsilon}_{J+M}^M\| \\ &\approx \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\ln|\nu_{Jj} - \nu_{(J+m)j}| - \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M-1}\ln|b_{jj}^{J+m}| \\ &= \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\ln|a_{jj}^J - a_{jj}^{J+m}| - \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M-1}\ln|b_{jj}^{J+m}| \\ &= E[\ln|a_{jj}^J - a_{jj}|] - E[\ln|b_{jj}|] \end{aligned} \tag{34}$$

From Eq. (34), it is clear that when the natural frequency is in the jth frequency group, i.e., when vibration in the jth substructural mode is dominant, the first-order approximate localization factor is given by Eq. (34), which amounts to considering only the jth substructural mode for each cantilever.

In Ref. 15, the method of transfer matrix was applied to determine the localization factors. For a system with S substructural modes,

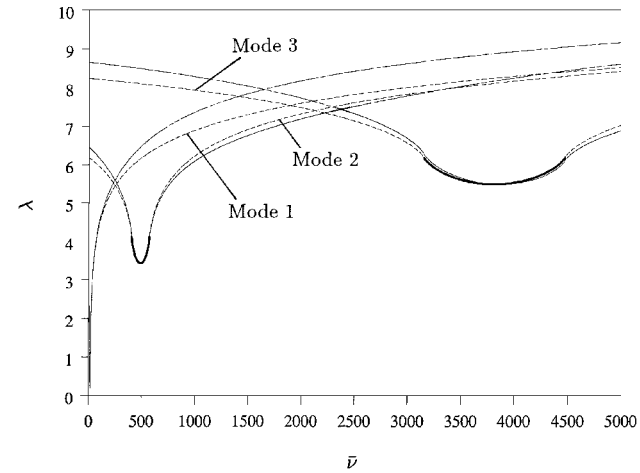


a) Three frequency groups

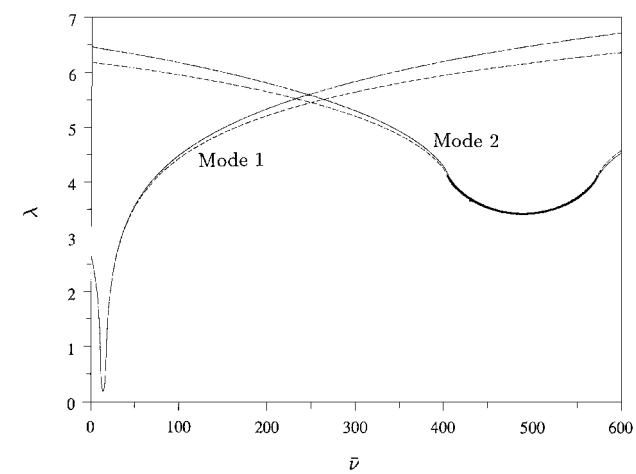


b) First two frequency groups

Fig. 4 Localization factors of a multicoupled one-dimensional system: $\bar{K}_i = 0.00125$, $\bar{k}_i = 0.005$, $\bar{m}_i = 1.0$, $\mu_{\overline{EI}} = 1.0$, and $\delta_{\overline{EI}} = 0.1$; ● method of perturbation, $N = 10^3$; – – –, λ_1; – – – –, λ_2; ——, λ_3, transfer matrix method, $N = 10^5$; – – –, only one substructural mode taken for each cantilever, transfer matrix method, $N = 10^5$.



a) Three frequency groups



b) First two frequency groups

Fig. 5 Localization factors of a multicoupled one-dimensional system: $\bar{K}_i = 0.125$, $\bar{k}_i = 0.5$, $\bar{m}_i = 1.0$, $\mu_{\overline{EI}} = 1.0$, and $\delta_{\overline{EI}} = 0.1$; ● method of perturbation, $N = 10^3$; – – –, λ_1; – – – –, λ_2; ——, λ_3, transfer matrix method, $N = 10^5$; – – –, only one substructural mode taken for each cantilever, transfer matrix method, $N = 10^5$.

the dimension of the transfer matrices is $2S \times 2S$; hence there are S positive Lyapunov exponents and S negative Lyapunov exponents. The smallest positive Lyapunov exponent is the localization factor. However, in Ref. 15, no explanations were provided as to the physical meanings of the first $S - 1$ positive Lyapunov exponents.

For weak coupling, in the jth frequency group, the jth substructural mode is dominant and the smallest positive Lyapunov exponent is approximately the Lyapunov exponent when only the jth substructural mode is taken for each cantilever. The transition of Lyapunov exponents from one frequency group to the other is of particular interest, which may provide physical insight to the meanings of all S positive Lyapunov exponents.

The localization factors of a cantilever–spring chain with $N = 1000$ cantilevers as given by Eq. (34) are plotted in Fig. 4 as solid dots for $\bar{K}_i = 0.00125$, $\bar{k}_i = 0.005$, $H = L$, $\bar{m}_i = 1.0$, $i = 1, 2, \ldots, N$, in which the first three substructural modes are taken for each cantilever. The nondimensional flexural stiffness $\overline{EI}_i$ of the ith cantilever is a uniformly distributed random number with mean $\mu_{\overline{EI}} = 1.0$ and coefficient of variation $\delta_{\overline{EI}} = 0.1$. In this case, $\bar{k} + 4\bar{K} = 0.005 + 4 \times 0.00125 = 0.01 \ll \hat{\beta}_1^4 (= 12.36)$. Hence, all cantilever modes may be regarded as weakly coupled. The localization factors are determined using Eq. (34), and the corresponding natural frequencies are determined using Eqs. (4) and (19), in which only the jth substructural mode is considered in the jth frequency group. The localization factors when only one substructural mode is taken for each cantilever are also evaluated using the method of transfer matrix for $N = 10^5$; the results are plotted in Fig. 4 as dashed lines. It is seen that these two results agree very well.

In Ref. 15, a numerical procedure was presented to evaluate all of the Lyapunov exponents of a multicoupled system, i.e., S substructural modes are taken for each cantilever. The algorithm is employed to determine the three positive Lyapunov exponents. As shown in Fig. 4, the smallest positive (the third) Lyapunov exponent agrees well, especially in the lower frequency groups, with the localization factors obtained considering only one substructural mode in each frequency group. Furthermore, the following is observed.

1) In the first frequency group, the first Lyapunov exponent is related to the third substructural mode, whereas the second Lyapunov exponent is related to the second substructural mode.

2) In the second frequency group, the first Lyapunov exponent is related to the third cantilever mode, whereas the second Lyapunov exponent is related to the first substructural mode.

3) In the third frequency group, the first Lyapunov exponent is related to the first cantilever mode, whereas the second Lyapunov exponent is related to the second substructural mode.

However, these relationships in the third frequency group are not as prominent as those in the first two frequency groups. Although there are differences in the numerical values of the Lyapunov exponents of a system and the localization factors of systems in which only one substructural mode is taken for each cantilever, the correspondences and the trends can be clearly seen.

It is noted that the results obtained earlier are not restricted to very small values of the coupling spring stiffnesses. In Fig. 5, the three positive Lyapunov exponents of a cantilever–spring chain with $N = 10^5$ are plotted for moderate values of the coupling spring stiffnesses $\bar{K}_i = 0.125$, $\bar{k}_i = 0.5$, $H = L$, $\bar{m}_i = 1.0$, $i = 1, 2, \ldots, N$, and

uniformly distributed $\bar{E}\bar{I}_i$ with $\mu_{\bar{E}I} = 1.0$ and $\delta_{\bar{E}I} = 0.1$. In this case, $\bar{k} + 4\bar{K} = 0.5 + 4 \times 0.125 = 1.0$, which is of the same order as $\hat{\beta}_1^4 = 12.36$, whereas $\bar{k} + 4\bar{K} \ll \hat{\beta}_2^4 (= 485.48)$. Hence, the first cantilever mode may not be considered as weakly coupled, and the method of regular perturbation may not be applied, whereas all other higher modes may be regarded as weakly coupled. When only one substructural mode is taken for each cantilever, the localization factors obtained by the method of transfer matrix are shown in Fig. 5 as dashed lines and those obtained using the regular perturbation method are shown as solid dots. It is seen that, even when some of the lower cantilever modes may not be considered as weakly coupled, similar relationships as noted earlier are also clearly observed.

V. Conclusions

In this paper, a general method of regular perturbation for linear eigenvalue problems was presented, in which the orders of the perturbation terms were extended to infinity. The method of regular perturbation was applied to study vibration mode localization in weakly coupled randomly disordered systems. For a mono-coupled one-dimensional cantilever–spring chain, the nondimensional natural frequencies and vibration modes are given by the eigenvalues and the corresponding eigenvectors of a linear eigenvalue problem of a symmetric tridiagonal form. By taking only the leading terms $\varepsilon^M_{j,j+M}$ in perturbation, first-order approximate results of the localization factors for weakly coupled systems were obtained. The results were compared with numerical results obtained using the method of transfer matrix; very good agreement was observed. For a one-dimensional cantilever–spring chain in which S substructural modes are taken for each cantilever, the system is of a symmetric block tridiagonal form. Approximate results for the localization factors were obtained using the method of perturbation. When the natural frequency is in the jth frequency group, the localization factor of the vibration mode is approximately given by the localization factor in which only the jth substructural mode is taken for each cantilever. The approximate results were compared with numerical results obtained using the method of transfer matrix; very good agreement was seen. The derivation of localization factors using the method of regular perturbation provided some insights into the phenomenon of vibration mode localization in weakly coupled one-dimensional systems.

In the companion paper, the method of perturbation is employed to study vibration mode localization in disordered two-dimensional cantilever–spring arrays.

Appendix: Equations of Motion of Multicoupled Cantilever–Spring Chains

Consider the free vibration of the multicoupled one-dimensional cantilever–spring chain as shown in Fig. A1. The mass of the ith mesh is assumed to be lumped on the surrounding cantilevers, i.e., the ith and the $(i + 1)$th cantilevers. For the ith cantilever, the kinetic energy and the strain energy are

$$T_{ci} = \frac{m_i}{2}\int_0^L \left[\frac{\partial}{\partial t} v_i(y, t)\right]^2 \mathrm{d}y$$

$$V_{ci} = \frac{EI_i}{2}\int_0^L \left[\frac{\partial^2}{\partial^2 y} v_i(y, t)\right]^2 \mathrm{d}y \tag{A1}$$

For the ith spring and mesh, the strain energy is given by

$$V_{si} = \int_0^L \frac{k_i}{2}[v_{i+1}(y, t) - v_i(y, t)]^2 \,\mathrm{d}y + \frac{K_i}{2}[v_{i+1}(H, t) - v_i(H, t)]^2 \tag{A2}$$

The deflection of the ith cantilever $v_i(y, t)$ may be written as

$$v_i(y, t) = \sum_{j=1}^{S} x_{i,j}(t)\phi_j(y) = \boldsymbol{x}_i^T(t)\boldsymbol{\phi}(y), \qquad i = 1, 2, \ldots, N \tag{A3}$$

where

$$\boldsymbol{x}_i(t) = \{x_{i1}, x_{i2}, \ldots, x_{iS}\}^T, \qquad \boldsymbol{\phi}(y) = \{\phi_1, \phi_2, \ldots, \phi_S\}^T$$

$$\phi_j(y) = \cosh\hat{\beta}_j\eta - \cos\hat{\beta}_j\eta - \hat{\alpha}_j(\sinh\hat{\beta}_j\eta - \sin\hat{\beta}_j\eta) \tag{A4}$$

$$\eta = y/L$$

and, for the first three cantilever modes, the parameters are

$$\hat{\alpha}_1 = 0.73409551, \qquad \hat{\beta}_1 = 1.87510407$$

$$\hat{\alpha}_2 = 1.01846732, \qquad \hat{\beta}_2 = 4.69409113$$

$$\hat{\alpha}_3 = 0.99922450, \qquad \hat{\beta}_3 = 7.85475744$$

The total kinetic energy of the cantilever–spring chain is

$$T = \sum_{i=1}^{N} T_{ci} = \sum_{i=1}^{N} \frac{m_i}{2}\int_0^L \left[\boldsymbol{x}_i^T(t)\boldsymbol{\phi}(y)\right]^2 \mathrm{d}y \tag{A5}$$

and the total strain energy is

$$\begin{aligned} V &= \sum_{i=1}^{N}(V_{ci} + V_{si}) \\ &= \sum_{i=1}^{N}\left(\frac{EI_i}{2}\int_0^L \left[\boldsymbol{x}_i^T(t)\boldsymbol{\phi}''^T(y)\right]^2 \mathrm{d}y \right. \\ &\quad + \frac{k_i}{2}\int_0^L \left\{\left[\boldsymbol{x}_{i+1}^T(t) - \boldsymbol{x}_i^T(t)\right]\boldsymbol{\phi}(y)\right\}^2 \mathrm{d}y \\ &\quad \left. + \frac{K_i}{2}\left\{\left[\boldsymbol{x}_{i+1}^T(t) - \boldsymbol{x}_i^T(t)\right]\boldsymbol{\phi}(y)\right\}^2\right) \end{aligned} \tag{A6}$$

Substituting Eqs. (A5) and (A6) into the Lagrange's equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\boldsymbol{x}}_i}\right) - \frac{\partial T}{\partial \boldsymbol{x}_i} + \frac{\partial V}{\partial \boldsymbol{x}_i} = \boldsymbol{0}, \qquad i = 1, 2, \ldots, N \tag{A7}$$

results, after some lengthy calculation, in the following equations of motion:

$$\begin{aligned} m_i L\ddot{\boldsymbol{x}}_i &- (k_{i-1}L\boldsymbol{I} + K_{i-1}\boldsymbol{\Phi}_H)\boldsymbol{x}_{i-1} - (k_i L\boldsymbol{I} + K_i\boldsymbol{\Phi}_H)\boldsymbol{x}_{i+1} \\ &+ \left[(EI/L^3)\hat{\beta}_4 + (k_{i-1} + k_i)L\boldsymbol{I} + (K_{i-1} + K_i)\boldsymbol{\Phi}_H\right]\boldsymbol{x}_i = \boldsymbol{0} \\ &\qquad i = 1, 2, \ldots, N \end{aligned} \tag{A8}$$

where

$$\hat{\beta}_4 = L^3\int_0^L \boldsymbol{\phi}''(y)[\boldsymbol{\phi}''(y)]^T \,\mathrm{d}y = \mathrm{diag}\{\hat{\beta}_1^4, \hat{\beta}_2^4, \ldots, \hat{\beta}_S^4\}$$

$$\boldsymbol{\Phi}_H = \boldsymbol{\phi}(H)\boldsymbol{\phi}^T(H)$$

Seeking solutions of the form $\boldsymbol{x}_i = (1/\sqrt{\bar{m}_i})\bar{\boldsymbol{u}}_i e^{i\omega t}$, $i = \sqrt{-1}$, substituting into Eqs. (A8), and nondimensionalizing the equations lead to a symmetric linear eigenvalue problem

$$\bar{\boldsymbol{A}}\bar{\boldsymbol{u}} = \bar{\nu}\bar{\boldsymbol{u}} \tag{A9}$$

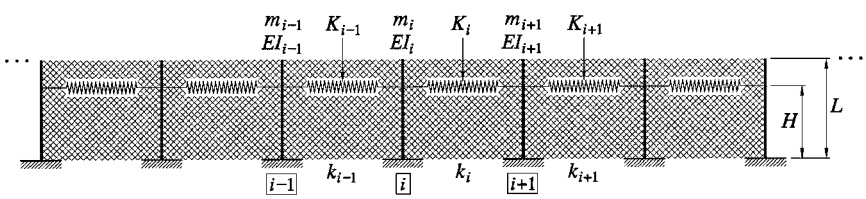


Fig. A1 Multicoupled one-dimensional cantilever–spring chain.

where

$$\bar{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1 & & & \\ \boldsymbol{B}_1 & \boldsymbol{A}_2 & \boldsymbol{B}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \boldsymbol{B}_{N-2} & \boldsymbol{A}_{N-1} & \boldsymbol{B}_{N-1} \\ & & & \boldsymbol{B}_{N-1} & \boldsymbol{A}_N \end{bmatrix}, \qquad \bar{\boldsymbol{u}} = \begin{Bmatrix} \bar{\boldsymbol{u}}_1 \\ \bar{\boldsymbol{u}}_2 \\ \vdots \\ \bar{\boldsymbol{u}}_N \end{Bmatrix}$$

$$\boldsymbol{A}_i = \frac{1}{\bar{m}_i}[\bar{E}I_i\hat{\beta}_4 + (\bar{k}_{i-1} + \bar{k}_i)\boldsymbol{I} + (\bar{K}_{i-1} + \bar{K}_i)\boldsymbol{\Phi}_H]$$

$$\boldsymbol{B}_i = -\frac{1}{\sqrt{\bar{m}_i\bar{m}_{i+1}}}(\bar{k}_i\boldsymbol{I} + \bar{K}_i\boldsymbol{\Phi}_H) \tag{A10}$$

$$\bar{\nu} = \frac{\omega^2}{\omega_0^2}, \qquad \omega_0^2 = \frac{EI}{mL^4}, \qquad \bar{k}_i = \frac{k_i L}{EI/L^3}$$

$$\bar{K}_i = \frac{K_i}{EI/L^3}, \qquad \overline{EI}_i = \frac{EI_i}{EI}, \qquad \bar{m}_i = \frac{m_i}{m}$$

If only the jth substructural mode is taken for each cantilever, the symmetric block tridiagonal matrix $\bar{\boldsymbol{A}}$ reduces to a symmetric tridiagonal matrix:

$$\bar{\boldsymbol{A}} = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{N-2} & \alpha_{N-1} & \beta_{N-1} \\ & & & \beta_{N-1} & \alpha_N \end{bmatrix}, \qquad \bar{\boldsymbol{u}} = \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_N \end{Bmatrix} \tag{A11}$$

where

$$\alpha_i = \frac{1}{\bar{m}_i}\left[\bar{E}I_i\beta_j^4 + \bar{k}_{i-1} + \bar{k}_i + (\bar{K}_{i-1} + \bar{K}_i)\phi_j^2(H)\right]$$

$$\beta_i = -\frac{1}{\sqrt{\bar{m}_i\bar{m}_{i+1}}}\left[\bar{k}_i + \bar{K}_i\phi_j^2(H)\right]$$

For system (A11) to be considered as weakly coupled, it is required that the magnitudes of the diagonal terms be much larger than those of the off-diagonal terms. Hence, if the first mode can be considered as weakly coupled, i.e., if $\hat{\beta}_1^4 \gg \bar{k} + 4\bar{K}$, because $\phi_1(L) = 2$, then the coupling for higher modes j becomes weaker with the increase of the number j of the modes because the term $\hat{\beta}_j^4$ increases rapidly with the increase of j. Depending on the system parameters, it is possible that the first $j < J$ modes may not be considered as weakly coupled, whereas the modes $j \geqslant J$ may be considered as weakly coupled if $\hat{\beta}_J^4 \gg \bar{k} + \bar{K}\phi_J^2(H)$.

Acknowledgment

The research for this paper was supported, in part, by the Natural Sciences and Engineering Research Council of Canada through Grant OGPO121355.

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